THE SOLUTION IN THE LARGE OF A CERTAIN SECOND ORDER ORDINARY LINEAR DIFFERENTIAL EQUATION OF RANK n

SENIOR THESIS

by

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I. INTRODUCTION

In order to introduce the investigations of the present paper, let us take for consideration the ordinary homogeneous linear differential equation of second order,

$$z \frac{d^2y}{dz^2} + k \frac{dy}{dz} + m^2 z^{2n-1} y = 0 \qquad (1.0)$$

Let the variable z be regarded as complex, as likewise the constants k and m. Let n be any positive integer. Then, the differential equation (1.0) will have in the language of Fuch's theory [1], a finite regular singular point at $z = 0$, and an irregular singular point at $z = \infty$, so that in all (1.0) has two singular points in the extended complex z-plane.

The indicial equation corresponding to $z = 0$ is found to be,

$$f(h) = h(h + k - 1) = 0 \qquad (1.1)$$

The two roots of this equation are $h = 0$ and $h = 1 - k$. Since k is complex their difference is not an integer. Then, the established theory assures that (1.0) will have two fundamental solutions of the form,

$$y_i(z) = z^{h_i} \sum_{n=o}^{\infty} c_{in} z^n \qquad i = 1,2 \qquad (1.2)$$

h_1 and h_2 being the two roots of (1.1). We know that each of these solutions will converge within a circle drawn about the point $z = 0$ and extending to the point $z = \infty$. At $z = \infty$, however, each becomes divergent. In particular, the behavior of y_i, $i = 1,2$ in the neighborhood of $z = \infty$ is not available, at least not without serious difficulties so far as Fuch's theory itself is concerned, since $z = \infty$ is an irregular singular point. However, from the established theory of linear differential equations it is known that in such a case, there will exist two linearly independent solutions $\bar{y}_i$, $i = 1,2$ which in the neighborhood of the point $z = \infty$ may be expressed asymptotically by means of normal series. It is known that there exist a set of connection coefficients, namely, Stoke's multipliers c_{ij} such that,

$$y_i(z) = \sum_{j=1}^{2} c_{ij}\, \bar{y}_j(z) \quad i = 1,2 \qquad (1.3)$$

Until some method of computing these coefficients is devised, the solution of the problem in the large cannot be considered as solved. It is to the solution of this problem in the large that the present thesis is addressed.

In order to do this, we shall look upon the question as that of determining the asymptotic developments of two functions y_i, $i = 1,2$ when these are regarded as defined merely by means of their Maclaurin development (1.2), that is without reference to the fact that they are the solutions of (1.0). In this manner we shall arrive independently at the fact that each is expressed when $|z|$ is large as a linear combination of the functions $\bar{y}_i$, $i = 1,2$ and we shall arrive also at the precise nature of this dependence.

II. SOLUTION OF THE DIFFERENTIAL EQUATION AT $z = 0$

Introducing the operator $U = z\frac{d}{dz}$, the differential equation,

$$z\frac{d^2y}{dz^2} + k\frac{dy}{dz} + m^2 z^{2n-1} y = 0 \text{ becomes} \tag{1.0}$$

$$f(U)y + z^{2n}g(U)y = 0 \tag{2.0}$$

where $f(U) = U(U + k - 1)$ and $g(U) = m^2$. The indicial equation corresponding to $z = 0$ is found to be,

$$f(h) = h(h + k - 1) = 0 \tag{2.1}$$

The roots of this equation are $h_1 = 0$ and $h_2 = 1 - k$. Since their difference is not a multiple of the skip number $2n$[2, page 418-423] the differential equation (1.0) has two solutions of the form,

$$y_i = z^{h_i}\sum_{r=o}^{\infty} c_{r_i} z^{2rn} \quad i = 1,2 \tag{2.2}$$

Substituting (2.2) in (1.0) and equating the coefficient of $z^{2n(r+1)+h_i}$ throughout, we get the two term recurrence relation,

$$c_{r+1,i}[2n(r+1) + h_i][2n(r+1) + h_i - 1] + k\, c_{r+1,i}[2n(r+1) + h_i] + m^2 c_{r,i} = 0$$

or

$$c_{r+1,i} = \frac{-m^2 c_{r,i}}{[2n(r+1) + h_i][2n(r+1) + h_i - 1+k]} \tag{2.3}$$

So, $$\left| \frac{c_{r+1,\, i}}{c_{r,i}} \right| = \left| \frac{-m^2}{[2n(r+1) + h_i][2n(r+1) + h_i - 1+k]} \right|$$

and $\lim_{r\to\infty} \left| \frac{c_{r+1,i}}{c_{r,i}} \right| = 0$. Hence (2.2) converges for all finite z by the ratio test. Also, iteration of (2.3) yields,

$$c_{r,i} = \frac{(-m^2)^r \; c_{oi}}{(2n)^{2r} \prod_{j=1}^{r} (j + \frac{h_i}{2n}) \prod_{j=1}^{r} (j + \frac{h_i}{2n} + \alpha)} \tag{2.4}$$

where c_{oi} is arbitrary, $\alpha = \frac{k - 1}{2n}$ and $i = 1,2$.

Taking $i = 1$ and $h_i = h_1 = 0$, we get,

$$c_{r1} = \frac{(-m^2)^r \; c_{o1}}{(2n)^{2r} \, \Gamma(r+1) \prod_{j=1}^{r} (j+\alpha)} \tag{2.5}$$

Similarly taking $i = 2$, $h_i = h_2 = 1 - k = -2n\alpha$, we get

$$c_{r2} = \frac{(-m^2)^r \; c_{o2}}{(2n)^{2r} \, \Gamma(r+1) \prod_{j=1}^{r} (j - \alpha)} \tag{2.6}$$

Here c_{o1} and c_{o2} are arbitrary. Choosing $c_{o1} = \frac{1}{\Gamma(1 + \alpha)}$ and $c_{o2} = \frac{1}{\Gamma(1 - \alpha)}$ we get,

$$c_{r1} = \frac{(-m^2)^r}{(2n)^{2r} \, \Gamma(r+1) \, \Gamma(r + \alpha + 1)} \tag{2.7}$$

$$c_{r2} = \frac{(-m^2)^r}{(2n)^{2r} \, \Gamma(r+1) \, \Gamma(r - \alpha + 1)} \tag{2.8}$$

Substituting this in (2.2) we get,

$$y_1 = \sum_{r=o}^{\infty} \frac{(\frac{im}{2n} z^n)^{2r}}{\Gamma(r + \alpha + 1)\,\Gamma(r + 1)}$$

$$y_2 = z^{1-k} \sum_{r=o}^{\infty} \frac{(\frac{im}{2n} z^n)^{2r}}{\Gamma(r - \alpha + 1)\,\Gamma(r + 1)} \tag{2.9}$$

which converge for all finite z .

$$\text{Let } \xi = \frac{im}{2n} z^n$$

$$\text{Then, } y_1 = \sum_{r=o}^{\infty} \frac{(\xi)^{2r}}{\Gamma(r + \alpha + 1)\,\Gamma(r + 1)} \tag{2.10}$$

$$y_2 = \beta \xi^{-2\alpha} \sum_{r=o}^{\infty} \frac{(\xi)^{2r}}{\Gamma(r - \alpha + 1)\,\Gamma(r + 1)}\,, \quad \text{where } \beta = (\frac{2n}{im})^{-2\alpha}$$

which converge for all finite ξ in the extended complex ξ-plane.

III. ASYMPTOTIC DEVELOPMENT OF SOLUTIONS AT $z = 0$

Now we are in a position to apply W. B. Ford's VIIth Theorem [3, p. 275]. Let $f(\xi)$ be the function of the complex variable ξ defined by the series

$$f(\xi) = \sum_{r=0}^{\infty} \frac{h(r)\xi^{2r}}{\Gamma(r + k_1)\,\Gamma(r + k_2)} \tag{3.0}$$

in which k_1, k_2 are constants (real or complex) and in which $h(r)$ may be regarded as a function $h(s)$ of the complex variable $s = x + iy$ and as such satisfies the two following conditions:

(a) $\dfrac{h(s)}{\Gamma(s + k_1)\,\Gamma(s + k_2)}$ is a single valued, analytic function of s throughout the finite s-space,

(b) $h(s)$ is such that when considered for values of s of large modulus lying in the right half plane $\mathcal{R}e(s) = x > x_o$, where x_o is some assignable number, it may be expressed in the form,

$$h(s) = c_o + \frac{c_1}{s+k_2} + \frac{c_2}{(s+k_2)(s+k_2+1)} + \cdots + \frac{c_{n+1} + \delta(s,n)}{(s+k_2)(s+k_2+1)\cdots(s+k_2+n)} \tag{3.1}$$

in which c_n are constants and $\lim_{s \to \infty} \delta(s,n) = 0 \quad n = 1,2,3\ldots$

Then $f(\xi)$ has the following asymptotic development

$$f(\xi) \sim \frac{e^{-2\xi}(-\xi)^{-p}}{2\sqrt{\pi}} \sum_{\ell=0}^{\infty} \frac{\bar{c}_\ell}{(-2\xi)^\ell} \; ; \quad \frac{\pi}{2} < \arg \xi < \frac{3\pi}{2} \qquad (3.2)$$

$$f(\xi) \sim \frac{e^{2\xi}\ \xi^{-p}}{2\sqrt{\pi}} \sum_{\ell=0}^{\infty} \frac{\bar{c}_\ell}{(2\xi)^\ell} \; ; \quad -\frac{\pi}{2} < \arg \xi < \frac{\pi}{2} \qquad (3.3)$$

in which $\bar{c}_\ell$ are determinate constants of which $\bar{c}_o = c_o$, where c_o is as in (3.1), where p is the constant given by $p = k_1 + k_2 - \frac{3}{2}$ and where it is understood that, if we take $\xi = \rho e^{i\psi}$, we take

$$(-\xi)^{-p} = e^{-p[\ell n\ \rho\ +\ i(\psi - \pi)]} \qquad 0 \leq \psi < 2\pi \qquad (3.4)$$

$$\xi^{-p} = e^{-p[\ell n\ \rho\ +\ i\ \psi]} \qquad 0 \leq \psi < 2\pi \qquad (3.5)$$

Moreover, if in condition (b) the quantity x_o may be regarded as an arbitrarily large negative number, the asymptotic development of $f(\xi)$ when $\arg \xi = \frac{\pi}{2}$, is that obtained when $\arg \xi = \frac{\pi}{2}$ in the following relation:

$$f(\xi) \sim -\sum_{\ell=1}^{\infty} \frac{h(-\ell)}{\Gamma(k_1-\ell)\ \Gamma(k_2-\ell)\xi^{2\ell}} \quad +$$

$$\frac{\xi^{-p}e^{2\xi}}{2\sqrt{\pi}} \sum_{\ell=0}^{\infty} \frac{\bar{c}_\ell}{(2\xi)^\ell} \; + \; \frac{(-\xi)^{-p}e^{-2\xi}}{2\sqrt{\pi}} \sum_{\ell=0}^{\infty} \frac{\bar{c}_\ell}{(-2\xi)^m} \qquad (3.6)$$

In all cases the interpretations (3.4) and (3.5) are to be used.

So by the above mentioned theorem y_1 and y_2 as given by (2.10) have the following asymptotic behavior for (ξ) large:

$$y_1 \sim \frac{e^{-2\xi}(-\xi)^{-p_1}}{2\sqrt{\pi}} \sum_{\ell=o}^{\infty} \frac{\bar{c}_\ell}{(-2\xi)^\ell} \qquad \frac{\pi}{2} < \arg \xi < \frac{3\pi}{2} \tag{3.7}$$

$$y_1 \sim \frac{e^{2\xi}(\xi)^{-p_1}}{2\sqrt{\pi}} \sum_{\ell=o}^{\infty} \frac{\bar{c}_\ell}{(2\xi)^\ell} \qquad -\frac{\pi}{2} < \arg \xi < \frac{\pi}{2} \tag{3.8}$$

$$y_1 \sim \frac{e^{-2\xi}(-\xi)^{-p_1}}{2\sqrt{\pi}} \sum_{\ell=o}^{\infty} \frac{\bar{c}_\ell}{(-2\xi)^\ell} + \frac{e^{2\xi}(\xi)^{-p_1}}{2\sqrt{\pi}} \sum_{\ell=o}^{\infty} \frac{\bar{c}_\ell}{(2\xi)^\ell} \qquad \arg \xi = \frac{\pi}{2} \tag{3.9}$$

where $p_1 = \alpha + \frac{1}{2}$ and $\bar{c}_o = 1$

Similarly

$$y_2 \sim \beta\xi^{-2\alpha} \frac{e^{-2\xi}(-\xi)^{-p_2}}{2\sqrt{\pi}} \sum_{\ell=o}^{\infty} \frac{\bar{d}_\ell}{(-2\xi)^\ell} \qquad \frac{\pi}{2} < \arg \xi < \frac{3\pi}{2} \tag{3.10}$$

$$y_2 \sim \beta\xi^{-2\alpha} \frac{e^{2\xi}(\xi)^{-p_2}}{2\sqrt{\pi}} \sum_{\ell=o}^{\infty} \frac{\bar{d}_\ell}{(2\xi)^\ell} \qquad -\frac{\pi}{2} < \arg \xi < \frac{\pi}{2} \tag{3.11}$$

$$y_2 \sim \beta\xi^{-2\alpha} \frac{e^{-2\xi}(-\xi)^{-p_2}}{2\sqrt{\pi}} \sum_{\ell=o}^{\infty} \frac{\bar{d}_\ell}{(-2\xi)^\ell} + \frac{\beta\xi^{-2\alpha} e^{2\xi}(\xi)^{-p_2}}{2\sqrt{\pi}} \sum_{\ell=o}^{\infty} \frac{\bar{d}_\ell}{(2\xi)^\ell} \qquad \arg \xi = \frac{\pi}{2} \tag{3.12}$$

where $p_2 = -\alpha + \frac{1}{2}$ and $\bar{d}_o = 1$

So,

$$y_1 \sim a_1\, e^{-2\xi}\, \xi^{-\alpha-1/2} \sum_{\ell=0}^{\infty} \frac{\bar{c}_\ell}{(-2\xi)^\ell} \qquad \frac{\pi}{2} < \arg \xi < \frac{3\pi}{2} \qquad (3.13)$$

$$y_1 \sim b_1\, e^{2\xi}\, \xi^{-\alpha-1/2} \sum_{\ell=0}^{\infty} \frac{\bar{c}_\ell}{(-2\xi)^\ell} \qquad -\frac{\pi}{2} < \arg \xi < \frac{\pi}{2} \qquad (3.14)$$

$$y_1 \sim a_1 e^{-2\xi} \xi^{-\alpha-1/2} \sum_{\ell=0}^{\infty} \frac{\bar{c}_\ell}{(-2\xi)^\ell} + b_1 e^{2\xi} \xi^{-\alpha-1/2} \sum_{\ell=0}^{\infty} \frac{\bar{c}_\ell}{(-2\xi)^\ell} \quad \arg \xi = \frac{\pi}{2} \quad (3.15)$$

where $a_1 = \dfrac{(-1)^{-\alpha-1/2}}{2\sqrt{\pi}}$, $b_1 = \dfrac{1}{2\sqrt{\pi}}$ and $\bar{c}_o = 1$

Similarly

$$y_2 \sim a_2\, e^{-2\xi}\, \xi^{-\alpha-1/2} \sum_{\ell=0}^{\infty} \frac{\bar{d}_\ell}{(-2\xi)^\ell} \qquad \frac{\pi}{2} < \arg \xi < \frac{3\pi}{2} \qquad (3.16)$$

$$y_2 \sim b_2\, e^{2\xi}\, \xi^{-\alpha-1/2} \sum_{\ell=0}^{\infty} \frac{\bar{d}_\ell}{(2\xi)^\ell} \qquad -\frac{\pi}{2} < \arg \xi < \frac{\pi}{2} \qquad (3.17)$$

$$y_2 \sim a_2 e^{-2\xi} \xi^{-\alpha-1/2} \sum_{\ell=0}^{\infty} \frac{\bar{d}_\ell}{(-2\xi)^\ell} + b_2 e^{2\xi} \xi^{-\alpha-1/2} \sum_{\ell=0}^{\infty} \frac{\bar{d}_\ell}{(2\xi)^\ell} \quad \arg \xi = \frac{\pi}{2} \quad (3.18)$$

where $a_2 = \dfrac{\beta}{2\sqrt{\pi}}\,(-1)^{\alpha-1/2}$ and $b_2 = \dfrac{\beta}{2\sqrt{\pi}}$, $\bar{d}_o = 1$ and

$\beta = \left(\dfrac{2n}{im}\right)^{-2\alpha}$

IV. ASYMPTOTIC STRUCTURE OF SOLUTIONS AT $z = \infty$

The point $z = \infty$, as already noted, is an irregular singular point of the differential equation (1.0). Since (1.0) is a differential equation of rank n, it would be advantageous to make the following transformation of the independent variable z.

$$\xi = \frac{im}{2n} z^n$$

Then the differential equation (1.0) becomes

$$\xi \frac{d^2y}{d\xi^2} + \left[\frac{n-1}{n} + \frac{k}{n} \right] \frac{dy}{d\xi} - 4\xi\, y = 0 \tag{4.0}$$

We are now in a position to apply Birkoff's theorem. G. D. Birkoff [4, pp. 463-468] has completely discussed the existence, form, range of the asymptotic solutions of the homogeneous linear differential equation

$$\frac{d^ny}{dz^n} + a_1(z) \frac{d^{n-1}y}{dz^{n-1}} + \cdots + a_n(z)y = 0$$

where the coefficient $a_r(z)$ are developable ($|z| > R$ = sufficiently large) either in convergent series, namely

$$a_r(z) = z^{rk}[a_{ro} + \frac{a_{r1}}{z} + \frac{a_{r2}}{z^2} + \cdots] \quad r = 1, 2, \ldots n \tag{4.1}$$

k being zero or a positive integer (the integer $k+1$ is called the rank of (4.0) at $z = \infty$). Birkoff's essential results may be summarised as follows:

Let, for the equation (4.0), the roots $\rho_1, \rho_2, \ldots \rho_n$ of the characteristic equation, i.e. the algebraic equation

$$\rho^n + a_{1\,o}\,\rho^{n-1} + \ldots + a_{n\,o} = 0 \tag{4.2}$$

be distinct from one another. Then there exists a set of fundamental solutions $\bar{y}_r (r = 1,2,\ldots n)$ of (4.0) in the neighborhood of $z = \infty$ developable asymptotically in forms

$$\bar{y}_r \sim e^{f_r(z)}\, z^{\rho_r^*} \sum_{j=o}^{\infty} \frac{a_{rj}}{z_j} \qquad r = 1,2,\ldots n$$

where $f_r(z)$ is a polynomial of degree $k+1$ in z, the coefficient of whose highest power of z is $\frac{\rho_r}{k+1}$ where ρ_r^* and a_{rj} are constants with $a_{r\,o} = 1$.

Applying the above mentioned Birkoff's theorem, the differential equation (4.0) has the characteristic equation $\rho^2 - 4 = 0 \quad \rho = \neq 2$.

Thus equation (4.0) will have two linearly independent solutions $\bar{y}_1$, $\bar{y}_2$, for which $|\xi|$ is large

$$\tilde{y}_1 \sim s_1$$

$$\tilde{y}_2 \sim s_2$$

where s_1 and s_2 are normal divergent series having the forms

$$s_1 = e^{-2\xi}\,\xi^{\lambda_1}\,[1 + \frac{(\ )}{\xi} + \frac{(\ )}{\xi^2} + \ldots]$$

$$s_2 = e^{2\xi}\,\xi^{\lambda_2}\,[1 + \frac{(\ )}{\xi} + \frac{(\ )}{\xi^2} + \ldots]$$

where λ_1 and λ_2 are constants whose values may be determined by formal substitutions of these series forms into (4.0) and are thus found to be

$$\lambda_1 = - \frac{1}{2} - \alpha$$

$$\lambda_2 = - \frac{1}{2} - \alpha$$

V. DETERMINATION OF STOKE'S MULTIPLIERS

Therefore, we have if $|\xi|$ is large, the solution y_j, $j = 1,2$ of (1.0) at $z = 0$ may be developed as follows:

$$y_1 \sim a_1 s_1 \qquad \frac{\pi}{2} < \arg\left[\frac{im}{2n} z^n\right] < \frac{3\pi}{2}$$

$$y_2 \sim b_1 s_2 \qquad -\frac{\pi}{2} < \arg\left[\frac{im}{2n} z^n\right] < \frac{\pi}{2}$$

and $$y_1 \sim a_1 s_1 + b_1 s_2 \qquad \arg\left[\frac{im}{2n} z^n\right] = \frac{\pi}{2}$$

where $$a_1 = \frac{(-1)^{-\alpha-\frac{1}{2}}}{2\sqrt{\pi}}, \qquad b_1 = \frac{1}{2\sqrt{\pi}}$$

Similarly

$$y_2 \sim a_2 s_1 \qquad \frac{\pi}{2} < \arg\left[\frac{im}{2n} z^n\right] < \frac{3\pi}{2}$$

$$y_2 \sim b_2 s_2 \qquad -\frac{\pi}{2} < \arg\left[\frac{im}{2n} z^n\right] < \frac{\pi}{2}$$

and $$y_2 \sim a_2 s_1 + b_2 s_2 \qquad \arg\left[\frac{im}{2n} z^n\right] = \frac{\pi}{2}$$

where $$a_2 = \frac{\beta}{2\sqrt{\pi}} (-1)^{+\alpha-\frac{1}{2}} \qquad b_2 = \frac{\beta}{2\sqrt{\pi}}$$

and $$\beta = \left(\frac{2n}{im}\right)^{-2\alpha}$$

VI. CRITIQUE

A successful attempt has been made in this paper to solve the differential equation (1.0) in the large, when $1 - k$ is a complex number. A close examination of (1.0) reveals the noteworthy fact that the results obtained in this paper hold good, even when k is a real number, provided that k is not equal to $mn + 1$ where m is an integer. No attempt has been made in this paper to solve (1.0) in the large under this hypothesis. In this case, one of the solutions y_i, $i = 1,2$ about z will be logarthmic in character and as a consequence, the evaluation of Stoke's multipliers and also the determination of the behavior of the solutions in the various sectors awaits a far more penetrating analysis to solve the problem in the large. One can attempt to complete this part of the analysis as well as to explore further the scope and limitations of the method that has been used to solve the problem in the large in this paper completely.

Besides these, there is one other direction of further possible research. That is, generalization in the direction of a differential equation of higher order than the second is evidently a possible field of study.

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